Finite element methods for boundary and interface problems

Roland Becker
University of Pau

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1 Subject

These lectures are concerned with the discretization of boundary value and interface problems by adaptive finite element methods. They are motivated by applications in solid and fluid mechanics, which provide a variety of different boundary conditions, either because of physical modeling or the need to limit the computational domain. In addition, many problems involve different physical materials, which lead to coupling of different partial differential equations by interface problems.

The mathematical theory of finite elements is mostly concerned with approximation properties, stability and error analysis, limiting the considered models to the simplest boundary conditions and generally excluding multi-physical couplings. Therefore, a standard first course on the mathematical theory of finite elements generally focusses on elliptic problems with homogenous Dirichlet data.

In recent time, considerable research has been made to develop numerical methods for multi-physics with an without meshes matching at the interfaces; naturally, non-matching meshes are at least interesting for problems with a priori unknown position of interfaces or dynamically moving interfaces.

These notes focus on finite element methods based on weak treatment of interface couplings. This can in principle be achieved by means of Lagrange multipliers. At least from a computational point of view, methods avoiding multipliers are preferable, since they avoid discretization of the multiplier spaces and do not require additional work for iterative solution of the resulting systems.

An important additional property of weak coupling is the possibility of weighting contributions from different partial differential operators involved in the system. Thereby it is possible to establish a discrete weak formulation which makes sense for the limit of a singularly perturbed problem, such as the convection-diffusion problem with vanishing viscosity parameter.

In 1971 Joachim Nitsche introduced [1] his method for the Poisson problem with non-homogenous Dirichlet conditions that avoids modification of the finite element spaces. Using the underlying idea in order to couple completely discontinuous finite element spaces on a cell-wise level goes back to [2] (and for the biharmonic problem to [3]). This led to the nowadays very popular discontinuous Galerkin finite element method [4, 5, 6, 7]. Another remarkable application of Nitsche's basic idea is the unfitted (or cut-) finite element method for interface problems [8], which turns out to be closely related to XFEM [9].

2 Plan of lectures

1. Nitsche's method
   (a) Elliptic problems
      i. Different finite element spaces
      ii. Nitsche's method, relation to multipliers
      iii. Elasiticty, Stokes
   (b) Hyperbolic problems
   (c) Contact problems
2. Interface problems
   (a) From domain decomposition to implicit interfaces
   (b) The elliptic interface problem : unfitted FEM an other methods
   (c) Tow-body contact
3. Adaptivity
   (a) Review of convergence theory
   (b) Residual estimators and flux recovery

References

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